MATH4010 Functional Analysis

Homework 6 suggested Solution

Question 1. Let X be a Hilbert space and let $T : X \to X$ be a bounded operator. Show that $\lim_{\lambda\to\infty} \left\|(T-\lambda)^{-1}\right\| = 0$, where $\lambda > \|T\|$.

Solution:

Let $\lambda > \|T\|$, then $(T-\lambda)$ is invertible. Note that

$$\frac{1}{\lambda}(T-\lambda)\left(1+\frac{1}{\lambda}T+(\frac{1}{\lambda}T)^2+\ldots+(\frac{1}{\lambda}T)^n\right) = (\frac{1}{\lambda}T)^{n+1} - I$$

Define $S_n(\lambda) := -\frac{1}{\lambda}\sum_{k=0}^{n}\left(\frac{1}{\lambda}T\right)^k$, then we have

$$(T-\lambda)S_n(\lambda) = I - (\frac{1}{\lambda}T)^{n+1} \tag{1}$$

Since $\|T\| < \lambda$, the sequence $\{S_n(\lambda)\}_n$ is absolutely convergent, which implies $\{S_n(\lambda)\}$ normly converges in Hirlbert space X. Denote $S(\lambda) = \lim_{n\to\infty} S_n(\lambda)$, then it follows from (1) that

$$(T-\lambda)S(\lambda) = I, \qquad S(\lambda)(T-\lambda) = I.$$

Thus $(T-\lambda)^{-1} = S(\lambda)$. Notice that

$$\begin{aligned}\|S(\lambda)\| = \| -\frac{1}{\lambda}\sum_{n=0}^{\infty}(\frac{1}{\lambda}T)^n\| &\le \frac{1}{\lambda}\sum_{n=0}^{\infty}\|(\frac{1}{\lambda}T)^n\| \\ &\le \frac{1}{\lambda}\sum_{n=0}^{\infty}\|\frac{1}{\lambda}T\|^n \\ &\le \frac{1}{\lambda}\frac{1}{1-\|\frac{1}{\lambda}T\|} \\ &= \frac{1}{\lambda-\|T\|}.\end{aligned}$$

Therefore $\lim_{\lambda\to\infty}\|(T-\lambda)^{-1}\|=0$, which follows by $\lim_{\lambda\to\infty}\frac{1}{\lambda-\|T\|}=0$.

Question 2. Using the notation given as in Question (1), let $f(t):=t^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0$ be a complex polynomials. Put $f(T):=T^n+a_{n-1}T^{n-1}+\cdots+a_1T+a_0I\in\mathcal{L}(X)$.

(i) Show that $f(T)$ is invertible in $\mathcal{L}(X)$ if and only if $\alpha\notin\sigma(T)$ for all roots α of f. (Hint: use the Fundamental Theorem of Algebra).

(ii) Let f be a polynomial as in Part (i). Show that if $\lambda_0\in\sigma(T)$, then $f(\lambda_0)\in\sigma(f(T))$.

Solution:

(i) By Fundamental Theorem of Algebra, $f(t)=\prod_{k=1}^n(t-\alpha_k)$, where $\alpha_1,\alpha_2,\ldots,\alpha_n\in\mathbb{C}$ are roots of f. Hence, we have $f(T)=\prod_{i=1}^n(T-\alpha_iI)$. Since X is Hilbert space, a complex number $\alpha\notin\sigma(T)$ iff $(T-\alpha)$ is linear isomorphism (*see* ***Remark 16.2***). Suppose $f(T)$ is invertible, then for any $1\le i\le n$,

$$I=f(T)^{-1}\cdot f(T)=f(T)^{-1}\cdot\prod_{k=1,k\neq i}^{n}(T-\alpha_kI)\cdot(T-\alpha_iI)\,; \tag{2}$$

$$I=f(T)\cdot f(T)^{-1}=(T-\alpha_iI)\prod_{k=1,k\neq i}^{n}(T-\alpha_kI)\cdot f(T)^{-1}. \tag{3}$$

It follows that $(T-\alpha_iI)$ is injective and surjective by (2) and (3), respectively. Therefore $\alpha_i\notin\sigma(T)$, for $1\le i\le n$.

Conversely, suppose $\alpha_i\notin\sigma(T)$ for $1\le i\le n$. Then $(T-\alpha_iI)$ is invertible and $(T-\alpha_iI)^{-1}\in\mathcal{L}(X)$. Notice that

$$f(T)\cdot\prod_{i=1}^n(T-\alpha_iI)^{-1}=\prod_{i=1}^n(T-\alpha_iI)\prod_{i=1}^n(T-\alpha_iI)^{-1}=I;$$

$$\prod_{i=1}^n(T-\alpha_iI)^{-1}\cdot f(T)=\prod_{i=1}^n(T-\alpha_iI)^{-1}\prod_{i=1}^n(T-\alpha_iI)=I.$$

Therefore $f(T)$ is invertible and $f(T)^{-1}=\prod_{i=1}^n(T-\alpha_iI)^{-1}$.

(ii) Let f be a polynomial as in Part (i) and $\lambda_0\in\sigma(T)$. Define $g(t)=f(t)-f(\lambda_0)$. From *2(i)* what has already been proved, we see that $g(T)$ is not invertible. Notice that $g(T)=f(T)-f(\lambda_0)I$, we have $f(\lambda_0)\in\sigma(f(T))$.